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A DIVISION OF THE RAMO-WOOLDRIDGE CORPORATION

THE EXPANSION OF A RAREFIED GAS INTO A VACUUM

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ABSTRACT

The expansion of an initially uniform gas sphere into a vacuum is described. The analysis proceeds on the basis of free molecular flow. It is demonstrated that collisions will not seriously modify the expansion process as long as the initial mean free path of a molecule within the sphere is much greater than the initial diameter of the sphere.

Expressions are derived for the momentum and energy flow associated with the expansion of such a sphere.

The expansion of a gas which fills half space and also of a gas initially of the form of a uniform cylinder are described.

The interaction of an expanding gas cloud with a specularly reflecting surface is determined.

The interaction of an expanding gas cloud with a diffusely reflecting surface is considered.

I INTRODUCTION

Let us consider a rocket vehicle operating in interplanetary space. The vehicle's rocket motor may be operating under steady-state thrust or sporadically in short bursts, either for orientation or for braking purposes. We are concerned with the phenomena that may be associated with the release of exhaust gases from the vehicle. For example: the released gases may by their very expansion into the vacuum react on the vehicle. The release gases may also, under the influence of solar radiation, fluoresce and ionize. Furthermore some of the ions so produced, or already present in the exhaust gases, will be trapped by the local magnetic fields.

In order to predict the magnitude and spatial extent of such effects, it is necessary, at the very least, to be able to predict the temporal evolution of the released gases.

In this paper we consider the expansion of a short burst of gas (a puff) into a vacuum. The continuum treatment of this apparently simple problem is amazingly difficult and by no means complete.[1]

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[1] See, for example, the treatment by J. B. Keller, Quarterly of Applied Mathematics, 14, (1956).

Therefore we confine ourselves to a treatment of this problem which can be carried through to completion and where the region of validity of the solution can be fairly well defined. The treatment we employ is that of the free molecular flow of gases; that is, we assume that the molecules in this puff of gas move to infinity without colliding with one another.

The solution is simple and proceeds in a straightforward manner. After we obtain the solution, we investigate the consequences of assuming a finite collision cross-section for the molecules. We find that the assumption of free molecular flow is not inconsistent with the assumption of the finite collision cross-section, provided the initial diameter of the puff is much less than the initial mean free path of the molecules.

II. THE FREE MOLECULAR EXPANSION OF A PUFF OF GAS INTO A VACUUM

We shall simulate our puff of gas by a spherical bubble of gas in containment by some sort of skin which can be made completely permeable at will. That is, the skin will retain the gas until a signal is given, at which point the skin ceases to exist and the gas is allowed to expand.

We shall furthermore assume that the gas molecules initially have a Maxwellian distribution of velocities at a temperature T, and do not collide with each other during the expansion (zero collision cross-section).

1. The Source Function for Free Molecular Expansion into a Vacuum

We assume that the skin bounding the sphere of gas disappears at $t = 0$.

Since the particles do not interact with each other, it is sufficient for the complete description of the problem to know the initial vector velocity of each particle. Then we can allow these particles to travel in straight lines after $t = 0$.

Let us take an element of volume $d\tau$ containing dN particles in the original sphere. Since, by hypothesis, these particles have a Maxwellian velocity distribution, we may immediately write the fraction f, of the dN particles with radial velocities ranging between v and $v + dv$. Thus:

$$f = 4\pi \left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta v^2} v^2 \, dv \tag{1}$$

where $\beta = m/2KT$ (m is the mass of the molecule, K is the Boltzman constant and T is the temperature of the gas).

At a time t, after the start of the expansion these particles with velocities v to $v + dv$ are contained between two concentric spheres having a center at $d\tau$ and radii $r = vt$ and $r + dr = (v + dv)\,t$. Therefore, the density of these particles, $d\rho$, is given by

$$d\rho = \frac{f\,dN}{4\pi dv\,t^3 v^2} \, . \tag{2}$$

Combining (2) with (1) and setting $v = r/t$, we obtain

$$d\rho = \left(\frac{\beta}{\pi}\right)^{3/2} \frac{e^{-\beta(r^2/t^2)}}{t^3} \, dN \, . \tag{3}$$

Equation (3) presents the density everywhere in time and space, due to the free expansion of a point Maxwellian ensemble of dN particles. It, therefore, deserves the title of "source function for free molecule expansion."

This source function is reminiscent of the source functions for diffusion and thermal conductivity. There is an important difference, however, and that is, time is to the first power in the exponent and to the 3/2 power in the denominator of these latter functions.

As can readily be demonstrated,

$$\int_0^\infty 4\pi r^2 \, d\rho \, dr = dN$$

independent of time; and, also, the time integral of the flux of particles through the surface of a surrounding sphere

$$\int_0^\infty 4\pi r^2 \frac{r}{t} \, d\rho \, dt \quad ,$$

is equal to dN, independent of r. These are necessary properties of a source function and are possessed by (3).

2. The Source Function for a Moving Source

The source function (3) can be generalized to take into account the motion of the source (i.e., the initial Maxwellian distribution is centered about a drift velocity V).

Let us treat the case of the moving observer, since the moving source can always be transformed to such a case. We have, according to Figure 1, an observer moving with velocity V. At $t = 0$ he is a distance r_o, from the source, s, and V makes an angle of θ with respect to the radius vector, r_o, drawn from the source.

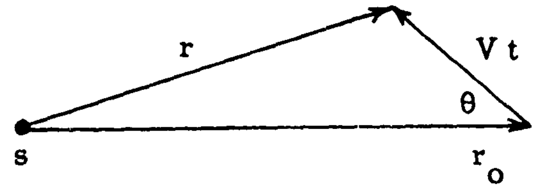

Figure 1

Now, the radial distance from the source will be given by

$$r = \sqrt{r_o^2 + V^2 t^2 - 2 V t r_o \cos \theta} \quad . \tag{4}$$

Thus since solution (3) is valid for all r and all time, the density encountered by the moving observer is:

$$d\rho = \left(\frac{\beta}{\pi}\right)^{3/2} dN \frac{e^{-\beta(r_o^2 + V^2 t^2 - 2 V t r_o \cos \theta)/t^2}}{t^3} \quad . \tag{5}$$

The situation as viewed by a stationary observer and moving source is thus:

$$d\rho = \left(\frac{\beta}{\pi}\right)^{3/2} dN \frac{e^{-\beta(r_o^2 + V^2 t^2 + 2 V t r_o \cos \theta)/t^2}}{t^3} \quad . \tag{5a}$$

When the source is in motion V can, in general, be a function of position so that (5a) is the source function for a body of gas, the individual parts of which are in relative motion (in the hydrodynamic sense) at $t = 0$.

3. The Uniform Gas Sphere in Free Expansion

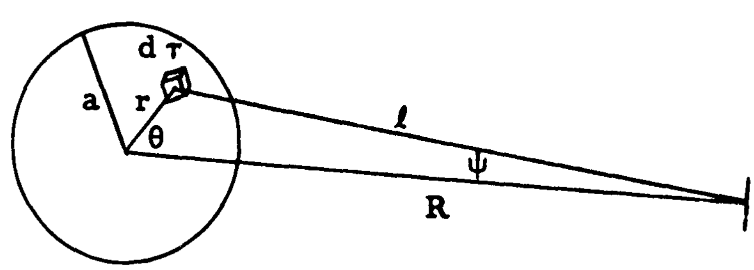

Figure 2

Using the source (3) we may now set up an expression, valid everywhere in space and time, for the density produced by the free expansion of an initially uniform sphere of gas with initial density ρ_o.

With reference to Figure 2:

$$\rho(R,t) = \int_0^a \int_0^\pi \rho_o \left(\frac{\beta}{\pi}\right)^{3/2} \frac{e^{-\beta \ell^2/t^2}}{t^3} \, 2\pi r^2 \sin\theta \, d\theta \, dr \tag{6}$$

where $\rho_o = dN/d\tau$. Since $\ell^2 = r^2 + R^2 - 2rR\cos\theta$ we may perform the tedious integration and obtain

$$\begin{aligned}\rho(R,t) = {} & \frac{\rho_o}{2}\left\{\operatorname{Erf}\left[(R+a)\frac{\sqrt{\beta}}{t}\right] - \operatorname{Erf}\left[(R-a)\frac{\sqrt{\beta}}{t}\right]\right\} \\ & + \frac{\rho_o t}{2R\sqrt{\beta\pi}}\left[e^{-(a+R)^2\beta/t^2} - e^{-(a-R)^2\beta/t^2}\right]\end{aligned} \tag{7}$$

where

$$\operatorname{Erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} \, dy \quad .$$

Equation (7) is simplified at either $R = 0$, the center of the sphere, or $R >> a$, where a is the initial extent of the sphere.

At $R = 0$ our solution becomes

$$\rho(0,t) = \rho_o \operatorname{Erf}\left(\frac{\beta^{1/2} a}{t}\right) - 2\rho_o \left(\frac{\beta}{\pi}\right)^{1/2} \frac{a}{t} e^{-\beta a^2/t^2} \tag{8}$$

and for large R, $(R >> a)$, $\ell \approx R$ and $\int_0^a \int_0^\pi \rho_o \, 2\pi r^2 \sin\theta \, d\theta = N$, the total number of particles in the original sphere. Thus for large R

$$\rho(R,t) = \left(\frac{\beta}{\pi}\right)^{3/2} \frac{e^{-\beta R^2/t^2}}{t^3} N \quad , \tag{9}$$

ρ having exactly the same form as the source function.

We can cast Eq. (9) into a form with more physical meaning by borrowing a description employed by J. B. Keller.[2] We assume the gas to have the specific

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[2] J. B. Keller, "On the Solution of the Boltzmann Equation for Rarefied Gases." Comm. Pure Appl. Math., 1, 275 (1948).

heat ratio γ and the velocity of sound in the original sphere is $C_o = \sqrt{\gamma\ (KT/m)}$. Then $\beta = \gamma/2C_o^2$ and therefore:

$$\rho(R,t) = \left(\frac{\gamma}{2\pi}\right)^{3/2} N \frac{e^{-\frac{\gamma}{2} R^2/(C_o t)^2}}{(C_o t)^3} . \tag{9a}$$

An inspection of Eq. (9a) readily reveals that the main body of the gas is contained in a sphere, the radius of which is expanding with the velocity $\sqrt{\frac{2}{\gamma}}\ C_o$.

4. <u>The Pressure and Impulse Due to an Expanding Gas Sphere</u>

The pressure at any point due to an expanding gas sphere may be determined by computing the momentum transport through a unit area per unit time. We set up a unit area as in Figure 2, perpendicular to the radius vector, R. The normal momentum flow per unit area per unit time due to each source element is $(d\rho) m v^2 \cos^2\psi$. Therefore, the pressure at R is:

$$P(R,t) = \int_0^a \int_0^\pi \rho_o 2\pi \left(\frac{\beta}{\pi}\right)^{3/2} m v^2 \cos^2\psi\, r^2 \sin\theta\, d\theta\, dr\, \frac{e^{-\beta(\ell^2/t^2)}}{t^3} . \tag{10}$$

where $v = \ell/t$, $\ell \cos\psi = R - r\cos\theta$, $\ell^2 = r^2 + R^2 - 2rR\cos\theta$.

Again, at large distances ($R \gg a$), $\ell \approx R$, $\cos^2\psi \approx 1$ and P becomes

$$P = M\left(\frac{\beta}{\pi}\right)^{3/2} \frac{R^2}{t^5} e^{-\beta(R^2/t^2)} = \left(\frac{\gamma}{2\pi}\right)^{3/2} \frac{M}{t^2} \frac{R^2}{(C_o t)^3} e^{-\frac{\gamma}{2} R^2/(C_o t)^2} \tag{10a}$$

where $M = Nm$, the total mass within the original sphere.

The momentum or impulse transmitted through the unit area is given by the time integral of (10). The total impulse per unit area is:

$$I = \int_0^a \int_0^\pi 2\pi \left(\frac{\beta}{\pi}\right)^{3/2} r^2 \sin\theta\, d\theta\, \rho_o m \frac{\cos^2\psi}{2\ell^2\beta^2}\, dr . \tag{11}$$

At large distances, this reduces to

$$I = \frac{M}{2R^2 \beta^{1/2} \pi^{3/2}} = \frac{M C_o}{R^2} \sqrt{\frac{2}{\gamma}} \frac{1}{2\pi^{3/2}} \quad . \tag{11a}$$

This last result is readily obtained by other means: Consider dN particles in a Maxwellian distribution. The momentum contained in those particles which have velocities v to $v + dv$, in a solid angle $d\Omega$ is

$$dI = m v^3 \left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta v^2} dv \, d\Omega \, dN \quad . \tag{12}$$

The average momentum in the direction of the infinitesimal solid angle $d\Omega$ is obtained by integrating over the velocity, thus

$$dI = \left(\frac{\beta}{\pi}\right)^{3/2} \frac{m \, dN}{2\beta^2} \, d\Omega \quad . \tag{12a}$$

Now instead of $d\Omega$, we can substitute its measure, the area, dA, subtended by this solid angle at the distance R, divided by R^2. Thus

$$\frac{dI}{dA} = \frac{m \, dN}{2\pi^{3/2} \beta^{1/2} R^2} \quad , \tag{12b}$$

equivalent to (11a).

5. Total Power and Energy Transmitted through Unit Area by an Expanding Gas Sphere

The flux of energy is given by $1/2 \, m v^2 \, (d\rho) \, v \cos\psi$ (refer to Fig. 2) where $v = \ell/t$. Thus, the power transmitted through a unit area is given by the following expression:

$$W = \frac{1}{2} m \left(\frac{\beta}{\pi}\right)^{3/2} 2\pi \int_0^a \int_0^\pi \rho_o \frac{\ell^3}{t^6} e^{-\beta(\ell^2/t^2)} \cos\psi \sin\theta \, d\theta \, r^2 \, dr. \tag{13}$$

Again at large distances ($R \gg a$), W reduces to

$$W = \frac{1}{2}\left(\frac{\beta}{\pi}\right)^{3/2} \frac{M}{t^6} R^3 e^{-\beta(R^2/t^2)} \tag{13a}$$

where $M = Nm$, the total mass in the initial sphere.

The total energy, E, passing through a unit area is obtained by integrating Eq. (13a) with respect to time. The result is

$$E = \frac{3}{16} \frac{M}{R^2 \pi \beta} \quad . \tag{14}$$

Thus the total energy crossing a large sphere about the original one is just $4\pi R^2$ times Eq. (14) or $\frac{3}{4}\frac{M}{\beta}$. This is a resonable result, for the average energy of the molecules originally in the sphere was $\frac{3}{4}\frac{m}{\beta}$ or $\frac{3}{2}$ KT. Thus the total energy in the sphere was $\frac{3}{4}\frac{M}{\beta}$ and this total energy is transported through the boundaries of the large bounding sphere, yielding Eq. (14).

III. THE CONSISTENCY CRITERIA FOR THE VALIDITY OF FREE MOLECULAR FLOW

We have up to this time assumed that the molecules do not interact with one another, that is, they have zero collision cross-section. We have described the expansion of a sphere of gas under these assumptions. We now ask the question: Under what criteria is this description valid for real molecules with finite collision cross-sections ?

We proceed to determine these criteria by the following method: Each molecule in the gas is to be endowed with a finite collision cross-section, but each molecule is artificially constrained to move in a straight line to infinity, even though it might collide with another molecule. The expansion of the gas is described according to the methods of Section II, but now we determine the number of collisions the average molecule encounters on its rigid and prescribed path to infinity. If this number is much less than one,

then we are correct in assuming that most of the molecules escape the sphere without suffering a collision. Consequently, the description of the expansion according to free molecular flow is valid.

1. <u>The Collisions Between one Particle from one Source and a Swarm of Particles from Another Source</u>

We first consider two point, particle sources, s_1 and s_2, and a particle of velocity V, emanating from s_1 at $t = 0$ and proceeding along a straight line. We shall assume that this particle does not collide with any other s_1 particles. The s_1 particle will pass through a swarm of particles emanating also at time $t = 0$ from s_2 (see Fig. 3).

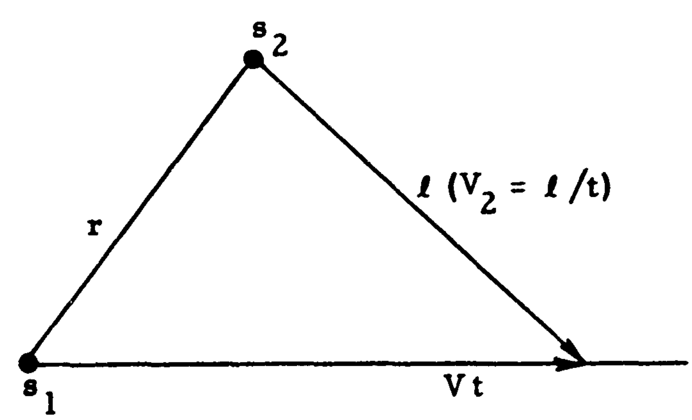

Figure 3

The collision frequency between the s_1 particle and the s_2 particles is given by $\rho_2 Q V_{12}$, where ρ_2 is the density of the s_2 particles encountered by the first particle in its path, Q is the collision cross-section and V_{12} is the relative velocity between the two particle species. When the expression $\rho_2 Q V_{12}$ is integrated over all time, the result will be the total number of collisions suffered by the s_1 particle in traveling an infinite distance from s_1.

The relative velocity, V_{12}, between particle s_1 and the particles it meets on its rectilinear path is just r/t, independent of the velocity of the s_1 particle. This is made evident by first considering those s_1 particles of zero velocity. The s_2 particles they meet have a velocity r/t. Now consider s_1 particles with velocity V in any direction. The s_2 particles that are going to be encountered must have this velocity, V, vectorially added to r/t. Thus, the relative velocity remains r/t.

Let us denote dn as the total number of collisions encountered by a particle of velocity V in the presence of source s_2. Thus:

$$dn = \int_0^\infty \rho_2 Q \frac{r}{t}\, dt \ , \tag{15}$$

but

$$\rho_2 = \rho_o\, d\tau \left(\frac{\beta}{\pi}\right)^{3/2} \frac{e^{-\beta(\ell^2/t^2)}}{t^3}$$

where $d\tau$ is the volume of the source s_2, ρ_o its density and $\ell^2 = r^2 + V^2t^2 - 2rVt\cos\theta$, [cf. Eq. (5)]. Thus:

$$dn = Qr\rho_o\, d\tau \left(\frac{\beta}{\pi}\right)^{3/2} \frac{V^3}{r^3} \int_0^\infty z^2 dz\, e^{-\beta V^2 (z^2+1)}\, e^{2\beta V^2 z\cos\theta} \ . \tag{15a}$$

2. <u>The Average Number of Collisions between Particles from one Source and a Swarm of Particles from Another Source</u>

Now the number of particles from s_1 with a velocity V to $V + dV$, at an angle θ and in a solid angle $2\pi\sin\theta\, d\theta$, is $2\pi(\beta/\pi)^{3/2}\sin\theta\, d\theta\, V^2 e^{-\beta V^2} dV$. We now weight (15a) with this factor and integrate over all angles and velocities, thus obtaining the average number of collisions encountered by all particles emanating from s_1 and proceeding to infinity.

We perform the θ integration first and obtain

$$\langle dn \rangle_\theta = \frac{Q\rho_o d\tau}{r^2}\left(\frac{\beta}{\pi}\right)^2 \int_o^\infty \int_o^\infty V^3 z e^{-\beta V^2} dz\, dv \left[e^{-\beta V^2 (z-1)^2} - e^{-\beta V^2 (z+1)^2}\right] \quad (16)$$

Next, the velocity integration

$$\langle\langle dn \rangle_\theta \rangle_V = \frac{Q dN}{2r^2} \frac{1}{\pi^2} \int_o^\infty z\, dz \left[\frac{1}{(z-1)^2+1} - \frac{1}{(z+1)^2+1}\right] \quad (17)$$

where $dN = \rho_o d\tau$ and, finally the z integration

$$\langle dn \rangle_{\theta,\, V} = \frac{Q dN}{2\pi r^2} \quad . \quad (18)$$

3. The Average Number of Collisions Encounted by a Particle in Escaping From a Uniform Spherical Gas Cloud

We now determine the average number of collisions encountered by particles from s_1 when s_1 is located in an initially uniform spherical cloud and s_1 is a distance R_o from the center.

We choose a polar coordinate system such that s_1 is on the polar axis, as in Figure 4.

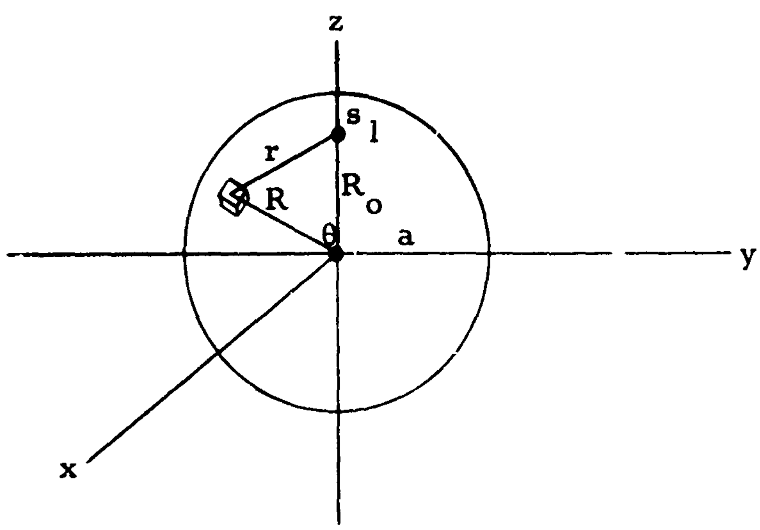

Figure 4

Then $r^2 = R^2 + R_o^2 - 2RR_o \cos\theta$ and therefore from (18)

$$\langle n \rangle_{\theta, V} = Q\rho_o \int_0^a \int_0^\pi \frac{R^2 \, dR \sin\theta \, d\theta}{R^2 + R_o^2 - 2RR_o \cos\theta} \quad . \tag{19}$$

Performing the θ integration first, we obtain

$$\langle n \rangle_{\theta, V} = \frac{Q\rho_o}{2R_o} \int_0^a R \, dR \, \ln \left(\frac{R+R_o}{R-R_o} \right)^2 \quad . \tag{19a}$$

Then the R integration:

$$\langle n \rangle_{\theta, V} = \frac{Q\rho_o}{2} \left[\frac{a^2}{R_o} \ln \left(\frac{a+R_o}{a-R_o} \right) + R_o \ln \left(\frac{a-R_o}{a+R_o} \right) + 2a \right] . \tag{19b}$$

Thus, when $R_o = 0$, $\langle n \rangle_{\theta, V} = 2Q\rho_o a$; when $R_o = a, \langle n \rangle_{\theta, V} = Q\rho_o a$ and when $R_o = \frac{1}{2} a, \langle n \rangle_{\theta, V} \approx \frac{7}{4} Q \rho_o a$.

Since $\rho_o Q$ is the reciprocal of the mean free path and $2a$ is the diameter of the sphere, we see that the requirements for neglecting collisions are precisely what we would expect, namely, that the diameter of the sphere be much less than the mean free path of the molecules, or equivalently that $\rho_o \ll \frac{1}{2Qa}$. Since the cross section is usually on the order of 10^{-16} cm^2 this means that $\rho_o \ll 10^{16}/2a$. Thus, for example, the expansion of a sphere initially with a diameter of one millimeter and with a particle density of 10^{16} per cm^{+3} may be treated as in the first two sections of this report.

APPENDIX I

OTHER TRANSIENT PROBLEMS INVOLVING THE FREE EXPANSION OF GASES INTO A VACUUM

Using the source function, Equation (3), we may solve other problems involving the free expansion of gases into a vacuum.

1. The Free Expansion of a Gas, Filling Half Space, into a Vacuum

We consider the problem, already solved by Keller[3] of the expansion of a gas for which $\rho = \rho_o$ for $x < 0$ and $\rho = 0$ for $x \geqslant 0$ at time $t \leqslant 0$.

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[3] cf. footnote 2.

First we determine the source function of a flat slab extending to infinity in the $\pm y$ and $\pm z$ directions but of thickness dx. Consider Figure I-1 where

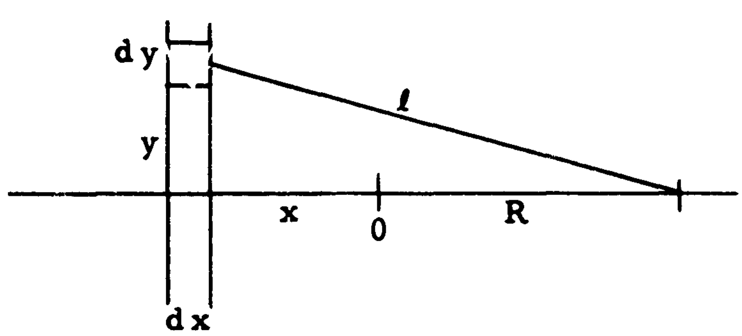

Figure I-1

we have the infinite slab of density ρ_o and an observer at distance R - · normal to the slab. The element of volume, $d\tau$, is $2\pi y\,dy\,dx$ and also $\ell^2 = (R-x)^2 + y^2$. Now using the source function (3), we find the density at R due to the initially uniform infinite slab.

$$d\rho = 2\pi\rho_o\left(\frac{\beta}{\pi}\right)^{3/2} \frac{1}{t^3}\int_0^{\infty} e^{-\beta(R-x)^2/t^2}\, e^{-\beta y^2/t^2}\, y\,dy\,dx$$

$$= \frac{\rho_o}{t}\left(\frac{\beta}{\pi}\right)^{1/2} e^{-\beta(R-x)^2/t^2}\,dx\,. \qquad (I.1)$$

The source function (I. 1) when integrated over x from $-\infty$ to 0 will give us the solution to the problem. Thus:

$$\rho(R, T) = \frac{\rho_o}{t}\left(\frac{\beta}{\pi}\right)^{1/2}\int_{-\infty}^{o} e^{-\beta(R-x)^2/t^2}\, dx \; . \tag{I. 2}$$

Let $\beta/t^2\ (R-x)^2 = \lambda^2$ and we obtain

$$\rho(R, T) = \frac{\rho_o}{\sqrt{\pi}}\int_{\sqrt{\beta}\frac{R}{t}}^{\infty} e^{-\lambda^2}\, d\lambda \quad , \tag{I. 3}$$

the same result obtained by Keller. This solution, as Keller points out, also describes the expansion of a semi-infinite gas in an infinite cylinder of rectangular cross-section with specularly reflecting walls.

2. <u>The Free Expansion of a line Source of Gas</u>

Consider a line source of gas of infinite length and of linear gas density, $dN/d\ell$. Then consider, Figure I-2, which illustrates the computation of the density at time t at a distance R from this line.

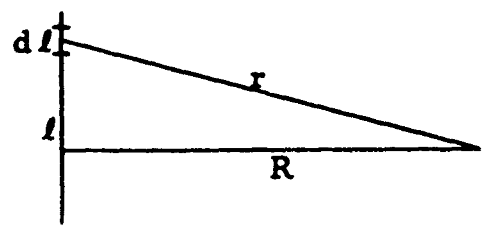

Figure I-2

Now $r^2 = \ell^2 + R^2$ and from Eq. (3)

$$d\rho = \frac{dN}{d\ell}\frac{\left(\frac{\beta}{\pi}\right)^{3/2}}{t^3}\int_{-\infty}^{\infty} e^{-\beta/t^2\ (R^2+\ell^2)}\, d\ell = \frac{dN}{d\ell}\left(\frac{\beta}{\pi}\right)\frac{e^{-\beta R^2/t^2}}{t^2} \; . \tag{I. 4}$$

We use this result immediately in the next section.

3. <u>The Free Expansion of a uniform cylinder of Gas, of Infinite Length, into a Vacuum</u>

We determine the source function of a cylindrical shell of infinite length. Consider Figure I-3

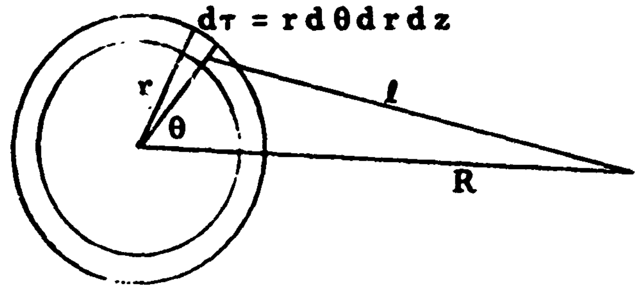

Figure I-3

The source function due to the generators of the cylinder is given by (I.4) and $dN/d\ell$ replaced by $\rho_o \, d\theta \, dr$ and R by ℓ. Therefore

$$d\rho = \int_0^{2\pi} \rho_o \frac{r\,dr}{t^2}\, d\theta \left(\frac{\beta}{\pi}\right) e^{-\beta/t^2\,(-2rR\cos\theta + r^2 + R^2)}$$

$$= 2\rho_o\beta \frac{r\,dr}{t^2}\, e^{-\beta/t^2\,(r^2+R^2)}\; J_o\left(i\,\frac{2rR\beta}{t^2}\right) . \qquad \text{(I. 5)}$$

The density due to an initially uniform expanding cylinder of radius a is therefore

$$\rho = 2\rho_o\beta \frac{e^{-\beta R^2/t^2}}{t^2} \int_0^a r\,dr\, e^{-\beta r^2/t^2}\, J_o\left(i\,2rR\,\frac{\beta}{t^2}\right) . \qquad \text{(I. 6)}$$

We need not concern ourselves with the difficult integrations required in (I. 6) provided we are at large distances ($R \gg a$) from the cylinder, for then $\ell = R$ and (I. 6) reduces to (I. 4) where $dN/d\ell = \rho_o \pi a^2$.

Equation (I. 6) may, by a simple transformation be reduced to

$$\rho = \rho_o \; e^{-\beta R^2/t^2} \int_o^{\beta a^2/t^2} dz\, e^{-z} \; J_o (i\, 2 \sqrt{y z}) \qquad \text{(I. 6a)}$$

where $y = R^2 \beta / t^2$.

The integral in (I. 6a) has been tabulated.[3, 4]

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[3] A. D. Wheelon and J. T. Robacher, A Table of Integrals Involving Bessel Functions, The R-W Corp., 1954, p. 57 Integral No. 3. 108.

[4] Math Tables and Aids to Comp. 6, 40, 1952.

APPENDIX II

THE INTERACTION OF AN EXPANDING GAS CLOUD WITH A SURFACE

1. Specular Reflection

a. A Source of Distance, R, from an Infinite Plane

Let us assume we have a point source, s, a distance, R, from an infinite plane. At $t = 0$, the gas molecules are emitted and travel radially. Those particles that strike the plane are assumed to be reflected specularly.

We determine the source function for this problem. Consider Fig. II-1

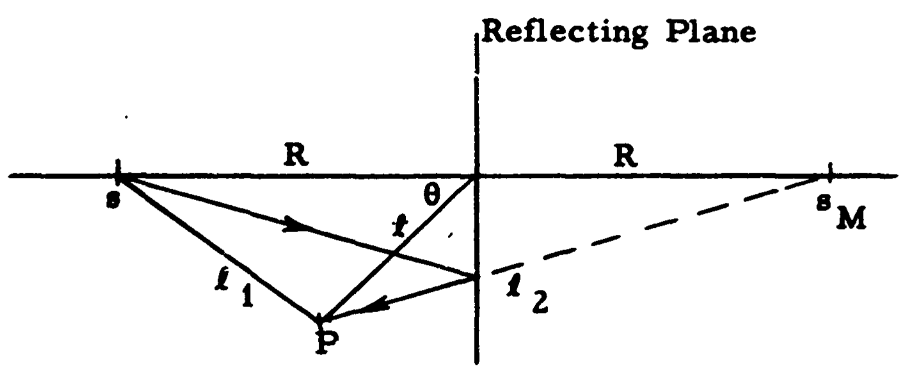

Figure II-1

It is obvious that half of the particles emitted by s are reflected from the plane and upon reflection appear to emanate from s_M, the mirror image. Because of this symmetry, the distance traveled by a particle from s to the plane to a point, P, is exactly the same in magnitude as ℓ_2, the length of a ray from s_M to P.

Now the density observed at a point, P, on the source side is composed of two parts, one due to the original source with a source strength d N, the other due to the mirror image, s_M, with the same source strength. Thus

$$d\rho = dN\left(\frac{\beta}{\pi}\right)^{3/2} \frac{e^{-\beta(\ell_1^2/t^2)}}{t^3} + dN\left(\frac{\beta}{\pi}\right)^{3/2} \frac{e^{-\beta(\ell_2^2/t^2)}}{t^3} \qquad \text{(II. 1)}$$

Now ℓ_1 and ℓ_2 may be written in terms of R, θ and ℓ, as in Fig. II-1, but we shall not do this here. Equation (1) may be used as a source function to describe the evolution of any gas cloud of any initial configuration in front of a plane.

The method of images employed here may be extended to more complicated cases of a source expanding in front of several intersecting planes. But in such cases, as in the equivalent electrostatic case, multiple images must be considered.

b. A Finite Source at the Center of a Sphere

A source is released at $t = 0$ at the center of an evacuated sphere of radius a. The particles travel radially until they strike the wall where they are reflected and then return to the center, traveling a distance $2a$ all together. The particles continue to travel back and forth from the center to the boundary. Our task is to predict the evolution of the gas density in time and space.

Let us assume that our source is a finite gas sphere of uniform density initially. It is at the center of a much larger sphere, so much larger that for most of the region within the sphere the evolution of the gas cloud may be described as though it came from a point source containing N particles [cf. Eq. (9)].

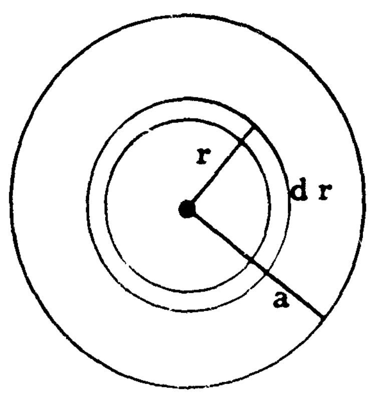

Figure II-2

We now ask, what particles will be found a distance, r, from the center between the two concentric spheres, r and $r + dr$, at time, t (see Fig. II-2)? The answer is: those particles with velocities $v_1 = r/t$, $v_2 = (2a - r)/t$, $v_3 = (2a + r)/t$, $v_4 = (4a - r)/t$, $v_5 = (4a + r)/t$, etc. Therefore we may write the density in space for all r much greater than the initial radius of the source sphere

$$\rho(r,t) = N\left(\frac{\beta}{\pi}\right)^{3/2} \frac{1}{t^3}\left[e^{-\beta(r^2/t^2)} + \frac{(2a-r)^2}{r^2} e^{-\beta/t^2 (2a-r)^2} + \frac{(2a+r)^2}{r^2} e^{-\beta/t^2 (2a+r)^2} + \frac{(4a-r)^2}{r^2} e^{-\beta/t^2 (4a-r)^2} + \ldots\right] . \quad (II.2)$$

2. <u>Diffuse Reflection</u>: <u>An infinitesimal source interacting with a diffusely reflecting surface</u>. Let us consider a source, s, of dN particles, released at time, $t = 0$. This source is some distance in front of a convex surface which acts as a diffuse reflector (see Fig. II-3).

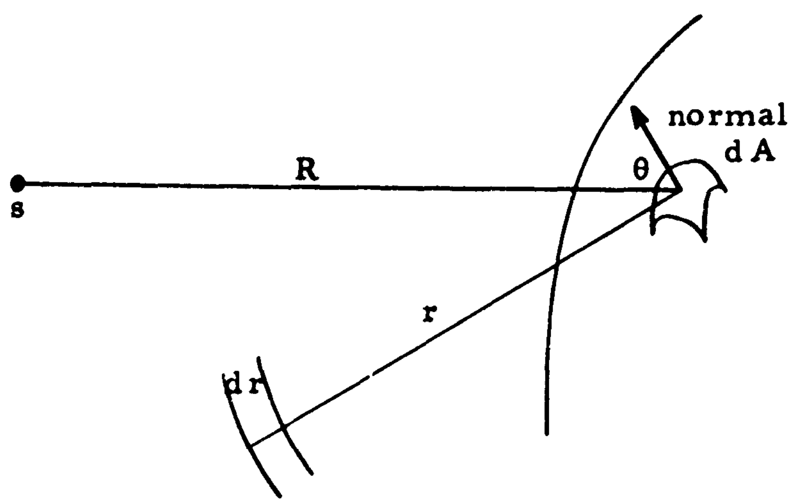

Figure II-3

Each element of area, dA, of this surface is assumed to reflect any incident particle into a random direction within a solid angle 2π. The reflected particles are also to have a Maxwellian velocity distribution with an associated temperature, T_A, not necessarily the temperature of the surface.

We are to determine the density of particles everywhere in space and time.

Now, the number of particles falling on d A per unit time and therefore emitted per unit time from d A is just:

$$\dot{N}(t) = \frac{R}{t} \, dA \cos\theta \, d\rho\,(t, R) \qquad \text{(II. 3)}$$

where $d\rho\,(t, R)$ is described by Eq. (3). We now proceed as in the development of Eq. (3).

During a time interval t' to $t' + \Delta t'$, $\dot{N}(t')\Delta t'$ particles are emitted from d A. Of these particles, the fraction, f, have velocities ranging between v and $v + \Delta v$. Now, by hypothesis f has the following form:

$$f = 4\pi \left(\frac{\beta_A}{\pi}\right)^{3/2} e^{-\beta_A v^2} v^2 \Delta v \; . \qquad \text{(II. 4)}$$

Thus at a time, t (greater than t'), after the release of the source, particles of number, $f \dot{N}(t')\Delta t'$, will be found within a hemispherical shell of radius, r, and thickness, Δr. It is obvious that $r = v\,(t - t')$ and $r + \Delta r = (v + \Delta v)\,(t - t' - \Delta t')$. The density of these particles at r is therefore:

$$d\rho_{A\,v} = \frac{2\dot{N}(t')\,\Delta t' \left(\frac{\beta_A}{\pi}\right)^{3/2} e^{-\beta_A v^2} \Delta v}{(t - t')^2 \left[v\,\Delta t' + \Delta v\,(t - t')\right]} \qquad \text{(II. 5)}$$

where $v = r/t - t'$ and $d\rho_{A\,v}$ is the density of particles reflected from the surface with velocity, v.

Now, the density of all reflected particles at r and at time, t, is obtained by counting particles with all velocities. The velocities for particles emanating from dA and found at r, t, range from a minimum of r/t to infinity, or equivalently, t' ranging from 0 to t. The density of reflected particles may therefore be obtained by

$$d\rho_A = \int_0^t 2\dot{N}(t')\,dt' \left(\frac{\beta_A}{\pi}\right)^{3/2} \frac{e^{-\beta_A \frac{r^2}{(t-t')^2}}}{(t-t')^3} \qquad \text{(II. 6)}$$

Substituting (II. 3) for $\dot{N}(t')$ and Eq. (3) for $d\rho$ we obtain the formidable integral:

$$d\rho_A = \left(\frac{\beta_A \beta}{\pi^2}\right)^{3/2} 2\int_0^t \frac{dt'}{t'^4 (t-t')^3} R\,dA \cos\theta\, dN \exp\left[-\frac{\beta R^2}{t'^2} - \frac{\beta_A r^2}{(t-t')^2}\right]. \qquad \text{(II. 7)}$$

Eq. (II. 7) can obviously be generalized to the case of a finite gas source. The density at r due to the whole reflecting surface may also be obtained from (II. 7) by integrating the expression over the surface. The density of all particles, $d\rho_t$, will be given by

$$d\rho_t = d\rho + d\rho_A \qquad \text{(II. 8)}$$

where $d\rho$ is the density of particles due to the free expansion of the source, s.

The solution of even the simplest problem (e.g., a point source in front of diffusely reflecting plane) presents formidable analytic difficulty in contrast to the case of specular reflection. No such solutions are presented in this paper.

The treatment of diffuse reflection from a concave surface requires consideration of multiple reflections and thus increases the difficulty of an already difficult problem.

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